Useful Equations

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

$$W = \int \vec{F}\cdot d\vec{s} = -\Delta U$$

$$t = \int \frac{ds}{v}$$

$$\vec{\ell} = \vec{r}\times\vec{p}$$

Rockets

$$m\dot{v} = -m v_{ex} + F^{ext}$$

$$v - v_0 = v_{ex}\ln(m/m_0)$$

Harmonic Oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Undamped:

$$x(t) = A\cos(\omega t - \delta)$$

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$$

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

Overdamped: $x = C_\pm\, e^{-\beta \pm \sqrt{\beta^2-\omega_0^2}}$

Critical: $x = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$

Underdamped: $x = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

Driven:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos\omega t$$

$$x(t) = A\cos(\omega t - \delta) + A_{tr}e^{-\beta t}\cos(\omega_1 t - \delta_{tr})$$

$$A^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$Q = \frac{\omega_0}{2\beta} = \pi\frac{\text{decay time}}{\text{period}}$$

Euler-Lagrange

$S = \int_{t1}^{t2}\mathcal{L}[q,\dot{q},t]\,dt$ stationary when $\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = 0$

$$\mathcal{L} = \text{T-U}$$

If $\frac{\partial\mathcal{L}}{\partial t} = 0$ then $\mathcal{L} - \sum_i \dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \text{constant}$

Two-Body Central Force

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$F_{cf} = \mu r\dot{\phi}^2 = \frac{\ell^2}{\mu r^3}$$

$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

Orbit Eqn: $u'' + u = -\frac{\mu}{\ell^2 u^2}F$

$[u = u(\phi)]$

Kepler: $r(\phi) = \frac{\alpha}{1+\varepsilon\cos\phi}$

$$\alpha = \frac{\ell^2}{k\mu},\; k = G\mu M$$

$$E = \frac{k^2\mu}{2\ell^2}(\varepsilon^2 - 1)$$

$$\tau^2 = 4\pi^2\frac{a^3\mu}{k}$$

Possibly Useful Integrals

$$\int\frac{dx}{(\alpha x+\beta)^2} = -\frac{1}{\alpha(\alpha x+\beta)} + C$$

$$\int\frac{dx}{(\alpha x+\beta)^{3/2}} = -\frac{2}{\alpha\sqrt{\alpha x+\beta}} + C$$

Physics 303 Fall 2015 Final Exam

1. An object glides to a stop in 1 dimension with a resistive force $f = -bv^{3/2}$.
Recall that $f = ma = m\dot{v}$.

a. How much time does it take for the object to come to rest from initial speed v_0?

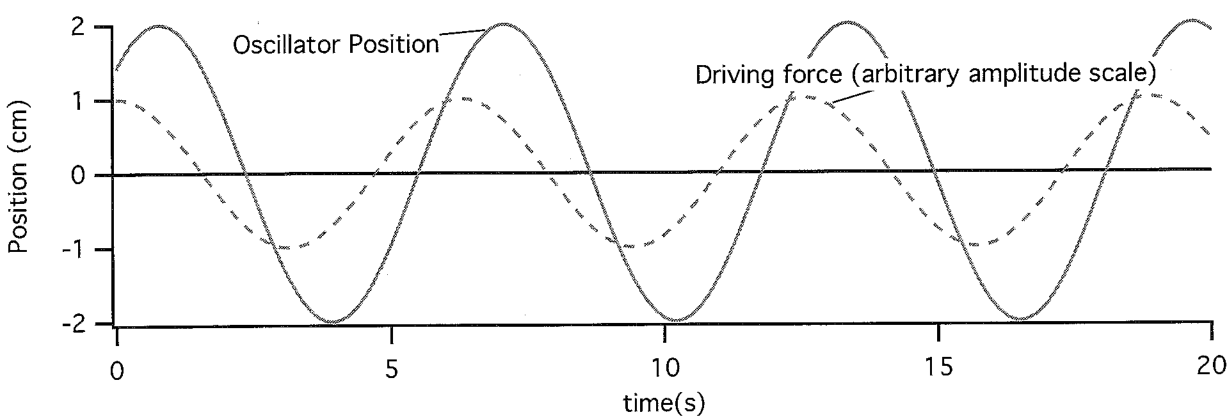
b. How far does the object travel before it comes to rest? To simplify your algebra, use the symbols $\alpha = b/m$ and $\beta = v_0^{-1/2}$.

2. The following graph shows the displacement of a driven, damped oscillator, driven by a cosine driving force at $\omega = 1\ s^{-1}$. The driving force has been applied for a long time.

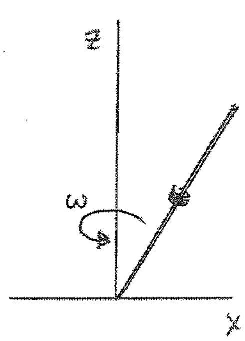
a. Is the driving frequency above, below, or at the resonant frequency?

b. The damping coefficient $\beta = 2\ s^{-1}$. What is the resonant frequency?

c. Sketch on the graph the approximate response if the driving force is removed at $t=3\pi/2$ seconds.



3. A straight wire is described by $z = \alpha x$. It rotates around the z-axis with constant angular velocity ω. A bead slides without friction on the wire. Gravity pulls toward $-z$.



a. Write down the Lagrangian for the bead, using x as the generalized coordinate.

b. Find the equation of motion.

c. Is there any equilibrium position for the bead? If so, where is it and is it stable?

d. Write down the complete solution to the equation of motion... it may contain one or more constants to be determined by initial conditions.

4. You wish to launch a rocket to study the inner solar system, collect samples of space dust and return to Earth. The idea is to put the spacecraft on an elliptical orbit that takes it close to the sun, and arrange for it rendezvous with the earth after exactly 1 (Earth) year. **The spacecraft will make 2 orbits while the Earth makes 1.** Assume the Earth's orbit is circular.

a. How close (in A.U. – the distance between the Earth and the Sun) will the spacecraft get to the sun? Hint: consider Kepler's law for orbital period.

b. What is the ellipticity ε of the spacecraft orbit?

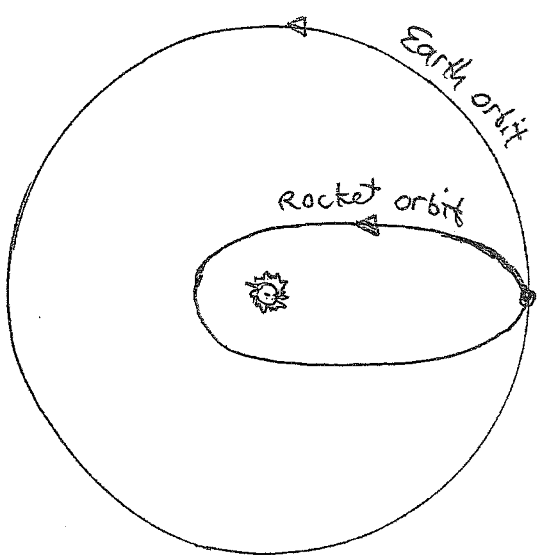
c. Write down the orbit equation ($r(\phi)$) for the orbit of the Earth and for the orbit of the spacecraft. Set them equal when spacecraft is at aphelion. What is the value of the scale factor c_{sc} for the spacecraft, compared to c_e for Earth?

d. By what factor must the angular momentum ℓ of the spacecraft change, if it is to change from being in a circular orbit around the sun (i.e. next to the Earth), to being in the elliptical orbit that takes it close to the sun?

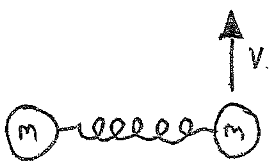
e. By what factor must the spacecraft velocity change?

f. Earth orbital velocity is 30 km/s. Rocket exhaust is typically 3 km/s. Ignoring the Earth's gravitation, what fraction of the rocket mass must be fuel for it to get into the correct orbit?

(Note: we'll need additional fuel at the rendezvous because we'll have to fire the rockets again to get into a circular orbit at R_e!)



5. Consider two identical masses *m* held by a massless spring, spring constant k, on a frictionless table. The masses are at rest and the spring is unstretched, length L.
One mass is given a small kick and thereby acquires a *small* "initial" velocity v (in the y direction).



a. What is the momentum of the center of mass, after the kick? Is it conserved?

b. What is the angular momentum of the system after the kick? Is it conserved?

c. Sketch the effective potential U_{eff} for the coordinate r, the separation between the masses.

d. For two masses with the same angular momentum, is there a stable circular orbit? If so, what is the *approximate* separation of the masses in this orbit? Hint: since r is close to L, you may write r-L = δ and keep only the lowest order terms.

e. What is the amplitude of the subsequent oscillations in the separation *r*, given that the initial kick is small?

f. Will the angular frequency of the oscillations in *r* be larger, smaller, or equal to $\sqrt{\frac{k}{\mu}}$?

You need to explain your reasoning in a sentence, not just guess.

g. What is the approximate angular frequency of the oscillations in r ? (Again, assume a small kick.)

h. Each mass is 2 kg, L=1 m, the spring constant k = 13 N/m. If the initial kick gives one mass a velocity of 1 m/s, sketch the "orbit" of this mass <u>in the center-of-mass frame</u>.